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INVARIANT SET STABILITY IN DISCRETE NON-LINEAR SYSTEMS†

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Sufficient conditions for local stability of invariant sets in non-linear discrete systems with time-varying parameters and difference inclusions are investigated. The results obtained are a step towards establishing an analogue of Lyapunov's First Method for the systems under consideration.

This work is related to investigations in robust stability theory [1, 2].

Most results for the stability of non-linear discrete inclusions are formulated using a Lyapunov function, and the problem of justifying an analogue of Lyapunov's First Method for these systems remains open. The use of the idea of this method to investigate non-linear systems with unknown time-varying parameters encounters some difficulties of principle. First, it is necessary to clarify the concept of "linearization" when applied to a multivalued right-hand side in the equation specifying the system behaviour. In general, this problem belongs to the field of multivalued mapping analysis and admits of several different approaches [3, 4]. Second, in the systems under consideration the equilibrium position is also, generally speak ing, not unique, and it makes no sense to talk about stationary solutions, but only about stationary sets of solutions, which need not be identical with the set of all equilibrium positions. Thus the system must be "linearized" in the neighbourhood of a set rather than a point, which hinders or renders impossible the use of what is now the well-developed concept of a linear approximation to a multivalued mapping in the neighbourhood of a point on its graph.

1. SETS OF STATIONARY SOLUTIONS AND INVARIANT SETS OF DISCRETE SYSTEMS

Consider a non-linear discrete dynamical system of the form

$$x_{i+1} = f(x_i, \xi_i), \quad \xi_i \in \Xi \tag{1.1}$$

Here $x_i \in R^n$ is the system phase-state vector, the $\xi_i \in R^m$ are unknown time-varying parameters, and $i = 1, 2, \ldots$ throughout. In all cases it is assumed that the function $f(\cdot, \cdot): R^n \times R^m \to R^n$ is continuous and the set $\Xi \subset R^m$ of allowed parameter values is compact.

As a generalization of (1.1) we consider the difference inclusion

$$x_{i+1} \in F(x_i) \tag{1.2}$$

where $F(\cdot)$ is a given multivalued mapping which is semicontinuous from above and takes compact values. In particular, (1.1) can be represented in the form (1.2) with $F(x) = f(x, \Xi) = \cup_{\xi \in \Xi} f(x, \xi)$.

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The special case of system (1.1) when the parameters are constant, i.e. $\xi_i = \xi \in \Xi$, is of particular interest. In this case (1.1) can be transformed to the form

$$\begin{aligned} &y_{i+1} = g(y_i), \quad y_i = (x_i, \xi_i) \in R^n \times R^m \\ &y_1 \in R^n \times \Xi, \quad g(y) = (f(x, \xi), \xi) \in R^n \times R^m \end{aligned} \tag{1.3}$$

In (1.3) the indeterminancy in the specification of the right-hand sides is turned into indeterminacy in the initial condition.

Let $y^0 = (x^0, \xi^0)$ be a stationary point of (1.3), $y^0 = g(y^0)$. Here it is obvious that x^0 is a stationary solution for system (1.1) with the condition $\xi_i = \xi^0$. We consider the set of all possible solutions of this form

$$X_0^s = \{x^0 \colon\ x^0 \in f(x^0, \Xi)\} \tag{1.4}$$

Under the given conditions X_0^s is a closed set, and when it is not empty its robust stability is of interest.

Definition 1.1. The closed set X is called robustly asymptotically stable (after Lyapunov) with respect to the stationary system (1.1) if for any $\varepsilon > 0$ a $\delta > 0$ exists such that for any solution $y_i = (x_i, \xi)$ of system (1.1) satisfying the conditions $d(x_1, X) < \delta$, $\xi \in \Xi$, the relations

$$d(x_i, X) < \varepsilon, \quad d(x_i, X) \to 0, \quad i \to \infty \tag{1.5}$$

hold, where $(d(x, X) = \inf\{\| x - z \| \colon z \in X\}$ is the distance from the point x to the set X.

We remark that the robust stability of the entire set X_0^s does not, in general, follow from the asymptotic stability (in projection onto R^n) of each of the stationary solutions. We will demonstrate the validity of this assertion by the following simple example.

Example 1.1. Let $n = 1$, $\Xi = \{0, 1\}$ and $x_{i+1} = \xi x_i + 1$.

If $\xi = 0$ we have $x^0 = 1$ as an asymptotically stable solution, whereas if $\xi = 1$ this system has no stationary solutions. Hence $X_0^s = \{1\}$. It is obvious that by Definition 1.1 this point is not robustly stable.

On the other hand, the following example shows that a stable set X_0^s does not necessarily consist of asymptotically stable stationary solutions.

Example 1.2. Suppose that $n = 1$, $\Xi = [-1, 1]$ and

$$x_{i+1} = \varphi(x_i) + \xi, \quad \varphi(x) = 4x(1 + |x|)^{-1}$$

One can show that in this case $X_0^s = [-(2+\sqrt{5}), (2+\sqrt{5})]$ is robustly stable. At the same time, when $\xi = 0$ the point $x = 0$ is an unstable position of equilibrium.

The set X_0^s consists of equilibrium positions of system (1.1) under the condition that the parameters ξ_i do not change over time. Here its construction is a relatively clear matter, at least from the formal algorithmic point of view. However, from Definition 1.1 it can be seen that for an arbitrary set X to be robustly stable the inclusion [5]

$$f(X, \Xi) = \bigcup_{x \in X, \xi \in \Xi} f(x, \xi) \subset X$$

must be satisfied.

Thus sets that are invariant under (1.1) can only be robustly stable when the condition that the parameters ξ_i should be stationary is not essential to their definition. With reference to this we investigate below the asymptotic stability of invariant sets of systems (1.1) and (1.2). It should be noted that the construction of non-trivial invariant sets is undoubtedly a more complicated problem, the discussion of which goes beyond the scope of this paper.

We will identify a special class of invariant sets which extends the concept of a stationary solution.

Definition 1.2. A closed set X_0 is a stationary set (SS) of system (1.1), (1.2) if the equality

(1.6)

is satisfied.

Relation (1.6) does not define a unique SS. In particular, a union of SSs is itself an SS. To eliminate this uncertainty we introduce the following definition.

Definition 1.3. An SS is called minimal if none of its proper subsets is an SS.

The problem of the existence of a non-empty SS can be solved using fixed point existence theorems for mappings defined in the space of compact (or closed) sets in R^n. We will not dwell on this range of problems, but formulate a single existence criterion which does not use fixed-point theorems and is based on a qualitative characteristic of dynamical systems.

Theorem 1.1. Suppose a non-empty set Z_0 exists such that all solutions of (1.2) that begin in this set remain in some compact set (i.e. the system is dissipative in Z_0 [6]). Then system (1.2) has a non-empty compact minimal SS.

We will only sketch the proof of this theorem. It can be divided into two stages. We first establish that under the conditions of the theorem a non-empty compact invariant set Y exists. To justify this statement it is sufficient to put

$$Y = \overline{\lim_{i\to\infty}} X_i, \quad X_{i+1} = F(X_i), \quad X_1 = Z_0$$

where $\overline{\lim}_{i\to\infty} X_i$ defines the upper limit of a sequence of compact sets in R^n. [4]

The subsequent stage of the proof is essentially non-constructive and is based on Zorn's lemma [7]. On the set of compact sets invariant under (1.2) we introduce the relation of ordering by inclusion. Every linearly ordered subset $\{Y_\alpha\}$ of this set has a minimal element $Y_0 = \cap_\alpha Y_\alpha \neq \varphi$. Hence a minimal compact invariant set exists which is an SS of system (1.2).

Remark 1.1. In the terminology that has been introduced, Maschler and Peleg [5] investigate, in particular, the question of whether all minimal SSs are single-point sets.

It should also be noted that expressions (1.4) and (1.6) lead to different sets which can be variously related to one another. We will illustrate the validity of this assertion by two one-dimensional examples.

Example 1.3. Suppose $x_{i+1} = x_i/2 + \xi_i$, $\Xi = \{-1, 0, 1\}$.
Then if $X_0^s = \{-2, 0, 2\}$ and $X_0 = [-2, 2]$, i.e. $X_0^s \neq X_0$, $X_0^s \subset X_0$.

Example 1.4. Suppose that

$$x_{i+1} = \varphi(x_i) + \xi_i, \quad \Xi = [-1, 1], \quad \varphi(x) = \begin{cases} (x-1)^2, & x > 1 \\ 0, & x \in [-1,1] \\ -(x+1)^2, & x < -1 \end{cases}$$

In this case $X_0^s = [-3, -2] \cup [-1, 1] \cup [2, 3]$ and $X_0 = [-1, 1]$. Therefore X_0 is a subset of X_0^s.

We also note that the set X_0^s of stationary solutions invariant under (1.1) is an SS. Investigation of the stability of SSs therefore enables one to draw conclusions on the stability of sets of stationary solutions. With reference to this, in the following we shall not use the condition that the parameters in (1.1) are stationary.

Definition 1.4. A non-empty, closed, invariant set X_0 is called strongly asymptotically stable (after Lyapunov) with respect to (1.1), ((1.2)), if for any $\varepsilon > 0$ a $\delta > 0$ exists such that all trajectories of system (1.1), ((1.2)) starting from a point x^0, $d(x^0, X_0) < \delta$ satisfy conditions (1.5).

2. STABILITY IN THE FIRST APPROXIMATION

We will study the sufficiency conditions for the stability of invariant sets of system (1.1) which can be formulated in terms characterizing the stability of a "linearized" system. The nature of the result obtained is not unexpected, but we are not aware of analogues of the theorem given below.

Theorem 2.1. Let X_0 be a compact, invariant set of system (1.1), and let $\mathrm{bd}X_0 = X_0 \setminus (\mathrm{int}X_0)$ be its boundary. We assume that in some neighbourhood of the boundary $\mathrm{bd}X_0$ the function $f(x, \xi)$ is continuously differentiable with respect to x, uniformly in ξ, so that

$$f(x+u,\xi) = f(x,\xi) + f_x'(x,\xi)u + \|u\| r(x,\xi,u) \tag{2.1}$$

where $r(x, \xi, u) \to 0$ uniformly in $(x, \xi) \in (\mathrm{bd}X_0) \times \Xi$ as $u \to 0$.

If the inequality $\| f_x'(x, \xi) \| < 1$ is satisfied for all $x \in \mathrm{bd}X_0$ and $\xi \in \Xi$, the set X_0 is strongly asymptotically stable relative to (1.1). (Here and below the matrix norm is assumed to be consistent with the vector norm on R^n.)

Proof. Let $\varepsilon > 0$ and

$$\mu = \max\{\| f_x'(x,\xi) \| : \ (x,\xi) \in (\mathrm{bd}\, X_0) \times \Xi\} \tag{2.2}$$

By the conditions of the theorem one can choose a number δ, $0 < \delta < \varepsilon$ such that

$$\mu + \| r(x, \xi, u) \| \leqslant \nu < 1 \tag{2.3}$$

for all $x \in \mathrm{bd}X_0$, $\xi \in \Xi$, $\| u \| < \delta$. We assume that $d(x^0, X_0) < \delta$ and that $\{x_i\}$ is an arbitrary trajectory of system (1.1) which begins at the point x^0, with an associated sequence of undefined parameters $\xi_i \in \Xi$, i.e.

$$x_{i+1} = f(x_i, \xi_i), \quad x_1 = x^0 \tag{2.4}$$

We construct a recursively defined sequence of points $s_i \in \mathrm{bd}X_0$ satisfying the inequalities

$$\| x_i - s_i \| \leqslant \nu^{i-1}\delta \tag{2.5}$$

which completes the proof of the theorem. For s_1 we choose an arbitrary point of $\mathrm{bd}X_0$ for which

$$\| x^0 - s_1 \| = d(x^0, X_0) < \delta$$

We assume that points $s_1, \ldots, s_l$ which lie on the boundary of the set X_0 and satisfy inequalities (2.5) have been constructed. By virtue of (2.1)–(2.5) the $z_l = f(s_l, \xi_l)$ satisfy the estimates

$$\| x_{l+1} - z_l \| = \| (f_x'(s_l,\xi_l)(x_l - s_l) + \| x_l - s_l \| r(s_l,\xi_l,x_l - s_l)) \| \leqslant$$
$$\leqslant (\| f_x'(s_l,\xi_l) \| + \| r(s_l,\xi_l,x_l - s_l) \|) \| x_l - s_l \| \leqslant$$
$$\leqslant (\mu + \| r(s_l,\xi_l,x_l - s_l) \|)\nu^{l-1}\delta \leqslant \nu^l\delta$$

The set X_0 is invariant, and therefore $z_l \in X_0$. Hence one can find a point $s_{l+1} \in \mathrm{bd}X_0$ for which the inequality $\| x_{l+1} - s_{l+1} \| = dx(x_{l+1}, X_0) \leqslant \| x_{l+1} - z_l \|$ is satisfied.

The theorem is proved.

Remark 2.1. Because the Lyapunov stability property does not depend on the choice of norm on R^n, and to every matrix norm there corresponds a vector norm that is consistent with it [8], Theorem 2.1 holds for an arbitrary choice of matrix norm.

Taking this remark into consideration, one can assert that Theorem 2.1 directly extends the standard Lyapunov asymptotic stability condition for an equilibrium position of a non-linear stationary system without undefined parameters. Indeed, according to Theorem 2.1 the point $x^0 \in R^n$, $x^0 = f(x^0)$ will be an asymptotically stable equilibrium position for the system $x_{i+1} = f(x_i)$ if some matrix norm of the Jacobian $f'(x^0)$ is less than unity. Because the spectral radius of an arbitrary matrix can be as closely approximated as desired by some norm of this matrix consistent with the vector norm on R^n [9], the conclusion of the theorem is equivalent to the fact that all the eigenvalues of the matrix $f'(x^0)$ lie inside the unit circle of the complex plane.

We consider some examples which illustrate the assertion.

Example 2.1. Let $n = 1$. We consider positive solutions of the system

$$x_{i+1} = f(x_i, \xi_i) = \xi\varphi(x_i), \quad x_1 > 0$$
$$\varphi(x) = 2x(|x| + 1)^{-1}, \quad \Xi = [\alpha, \beta], \quad (1/2) < \alpha < \beta$$

The positive asymptotically stable equilibrium positions of this system form the interval $X_0^\varsigma = [2\alpha - 1, 2\beta - 1]$. Indeed, for $x(\xi) = 2\xi - 1$ the relations

$$x(\xi) = f(x(\xi), \xi), \quad f_x'(x(\xi), \xi) = (2\xi)^{-1} < 1$$

hold.

Under the given assumptions X_0^ς is an SS, and by Theorem 2.1 its strong stability is ensured if the inequality

$$\max_{\xi\in[\alpha,\beta]} \max\{f_x'(2\alpha - 1, \xi), f_x'(2\beta - 1, \xi)\} = f_x'(2\alpha - 1, \beta) = (1/2)\beta\alpha^{-2} < 1$$

is satisfied.

Example 2.2. Consider an impulsive implementation of a simple servomechanism

$$x^{\cdot}(t) = k(u(t) - x(t)), \quad t \geqslant 0 \tag{2.6}$$

where $u(\cdot)$ is the input and $x(\cdot)$ is the output signal, and $k > 0$ is the gain. We assume that the rate of change of the input signal $u^{\cdot}(t) = \xi(t)$ is not known exactly and can vary between the limits $-\delta$ and δ, $\delta > 0$. If measurements of the input signal are made at discrete times t_i, then the discrete implementation of (2.6) leads to the difference equation

$$x(t_{i+1}) = -k(t_{i+1} - t_i)(x(t_i) - u(t_i)) + x(t_i)$$

Introducing the error $e_i = x(t_i) - u(t_i)$, we obtain

$$e_{i+1} = (1 - k\Delta t_i)e_i + \xi_i\Delta t_i \tag{2.7}$$

where $(-\xi_i) \in [-\delta, \delta]$ is the mean rate of change of the input signal over the interval $[t_{i+1}, t_i]$, and $\Delta t_i = t_{i+1} - t_i$ is the length of this interval.

If $\Delta t_i = \tau = \text{const}$, we obtain

$$e_{i+1} = \kappa e_i + \xi_i\tau, \quad \kappa = 1 - k\tau, \quad \xi_i \in [-\delta, \delta] \tag{2.8}$$

The set E_0^ς of stationary solutions of (2.8) is identical with the interval $[-\delta/k, \delta/k]$, which is an invariant set when $\kappa \geqslant 1$. One can show that when $-1 < \kappa < 0$ the interval $E_0 = (\tau/2 - k\tau))[-\delta, \delta]$ is a minimal SS of system (2.8). By Theorem 2.1 these sets are asymptotically stable when $k\tau \leqslant 1$ and $k\tau < 2$ respectively. When $\kappa \leqslant -1$ system (2.8) has no compact invariant sets.

We will now assume that pulse-frequency modulation is used, for which the length of the interval $[t_i, t_{i+1}]$ is a function of the error e_i, i.e. $\Delta t_i = \tau(e_i)$. In this case (2.7) is transformed to the form

$$e_{i+1} = f(e_i, \xi_i) = (1 - k\tau(e_i))e_i + \xi_i \tau(e_i) \tag{2.9}$$

The set of stationary solutions E_0^s does not change when passing from (2.8) to (2.9), although investigation of its invariance conditions requires very cumbersome calculations. Without going into the detail of these calculations, which can be performed for specific functions $\tau(\cdot)$, we note that on the basis of Theorem 2.1 the asymptotic stability of an invariant set in the form of an interval $[-\gamma, \gamma]$ can be ensured by the condition

$$\max_{\xi \in [-\delta, \delta]} |f_e'(\pm\gamma, \xi)| < 1 \tag{2.10}$$

In particular, if the function $\tau(e)$ is symmetric about zero, increases monotonically for negative e, and $1 - k\tau(e) \geqslant 0$ for all e, then from (2.9) and (2.10) we obtain the inequality

$$\tau'(-\gamma) = -\tau'(\gamma) < k\tau(\gamma)/(\gamma k + \delta)$$

3. LINEARIZATION OF DIFFERENTIAL INCLUSIONS

We will obtain an analogue of Theorem 2.1 for the difference inclusion (1.2). Let S be a closed unit sphere in R^n, and let $h(X, Y)$ be the Hausdorff distance between the sets X, $Y \subset R^n$.

Theorem 3.1. Let X_0 be a stationary set of (1.2) and suppose that the condition

$$\lim_{\delta \searrow 0} \sup \delta^{-1} h(X_0, F(X_0 + \delta S)) < 1 \tag{3.1}$$

is satisfied.

Then the set X_0 is strongly asymptotically stable with respect to (1.2).

Proof. It follows from (3.1) that $\omega < 1$ and $\delta_0 > 0$ exist such that

$$h(X_0, F(X_0 + \delta S)) \leqslant \omega\delta \tag{3.2}$$

for all $0 < \delta < \delta_0$. If $\{x_i\}$ is any solution of (1.2), and for some number j and $\delta_j < \delta_0$ we have the inclusion $x_j \in X_0 + \delta_j S$, then it follows from (1.2) and (3.2) that

$$x_{j+1} \in F(x_j) \subset F(X_0 + \delta_j S) \subset X_0 + \omega\delta_j S$$

Thus any solution of the difference inclusion (1.2) with initial condition $x_1 = x^0$, $d(x^0, X_0) < \delta \leqslant \delta_0$ satisfies the inclusion

$$x_i \in X_0 + \delta_i S, \quad \delta_{i+1} = \omega\delta_i, \quad \delta_1 = \delta$$

as was required.

Theorem 3.1 directly extends Theorem 2.1. Indeed, under the conditions of Theorem 2.1 the inclusion

$$\begin{aligned} f(X_0 + \delta S, \Xi) &= \bigcup_{\substack{u \in S, \xi \in \Xi \\ x_0 \in X_0}} f(x_0 + \delta u, \xi) = \\ &= \bigcup_{\substack{u \in S, \xi \in \Xi \\ x_0 \in X_0}} \{f(x_0, \xi) + \delta f_x'(x_0, \xi)u + \delta r(x_0, \xi, u)\} \subset \bigcup_{x_0 \in X_0} f(x_0, \Xi) + \delta\nu S = X_0 + \delta\nu S \end{aligned}$$

holds, from which (3.1) follows.

Remark 3.1. It follows from the proof of Theorem 3.1 that the Hausdorff distance in (3.1) can be replaced by the function

$$h_+(X, Y) = \inf\{t \geqslant 0:\ Y \subset X + tS\}$$

giving the deviation of the set Y from the set X.

Remark 3.2. In Theorem 3.1 the compactness of the set X_0 is not required in general, but for unbounded sets X_0 condition (3.1) is very restrictive.

It would be natural to connect the assertion of Theorem 3.1 with some construction defining the derivative of a multivalued map (MM) $F(\cdot)$. Concepts of derivatives based on an approximation to the graph of the MM at a point are not appropriate in this case for two reasons. The first is associated with the fact that, generally speaking, we are investigating the behaviour of a discrete system in the neighbourhood of an entire family of solutions rather than of a single (stationary) solution. The second reason is due to the tendency to investigate strong stability whereas the given constructions are obviously more suitable for obtaining weak stability conditions [10].

The following result, based on the concept of the strong derivative of an MM [11], is formally satisfactory.

Theorem 3.2. Suppose that the MM $F(\cdot)$ is continuous and has convex compact values, and that X_0 is a compact SS of (1.2). If the map $F(\cdot)$ is strongly differentiable at each point $x^0 \in \mathrm{bd}X_0$ in any direction $\upsilon \in S$, so that its derivative is continuous and its norm does not exceed some positive $\nu < 1$, then the set X_0 is strongly asymptotically stable with respect to (1.2).

Proof. The conditions of the theorem mean the following. For all $x^0 \in \mathrm{bd}X_0$ and $\upsilon \in S$ there are convex compact sets $F_+'(x^0, \upsilon)$ and $F_+'(x^0, \upsilon)$ such that

$$\begin{gathered} h(F_+'(x^0,\upsilon), F_-'(x^0,\upsilon)) \leqslant \nu \\ \lim_{\delta \searrow 0} \delta^{-1} h(F(x^0+\delta\upsilon)+\delta F_-'(x^0,\upsilon), F(x^0)+\delta F_+'(x^0,\upsilon)) = 0 \end{gathered} \tag{3.3}$$

We put

$$\rho(x^0,\upsilon,\delta) = h(F(x^0+\delta\upsilon)+\delta F_-'(x^0,\upsilon), F(x^0)+\delta F_+'(x^0,\upsilon))$$

Under the conditions of the theorem the function $\rho(\cdot,\ \cdot,\ \cdot)$ is continuous and

$$\rho_*(\delta) = \max_{x^0 \in X_0, \upsilon \in S} \rho(x^0,\upsilon,\delta)$$

satisfies the condition $\delta^{-1}\rho_*(\delta) \to 0$ as $\delta \to 0$. From (3.3) we obtain

$$F(x^0+\delta\upsilon)+\delta F_-'(x^0,\upsilon) \subset F(x^0)+\delta F_+'(x^0,\upsilon)+\rho(x^0,\upsilon,\delta)S \subset$$

$$\subset F(x^0)+\delta F_-'(x^0,\upsilon)+\delta(\nu+\delta^{-1}\rho_*(\delta))S$$

$$F(x^0)+\delta F_+'(x^0,\upsilon) \subset F(x^0+\delta\upsilon)+\delta F_-'(x^0,\upsilon)+\rho(x^0,\upsilon,\delta)S \subset$$

$$\subset F(x^0+\delta\upsilon)+\delta F_+'(x^0,\upsilon))+\delta(\nu+\delta^{-1}\rho_*(\delta))S$$

From this, using the stationarity of the set X_0 for all sufficiently small δ, we obtain the inclusions

$$F(X_0+\delta S) \subset X_0+\delta(\nu+\delta^{-1}\rho_*(\delta))S, \quad X_0 \subset F(X_0+\delta S)+\delta(\nu+\delta^{-1}\rho_*(\delta))S$$

which are equivalent to (3.1). The theorem is proved.

Example 3.1. Let X_0 be an SS with respect to the inclusion

$$x_{i+1} \in f(x_i) + \Xi$$

where $f(\cdot): R^n \to R^n$ is some continuously differentiable function, and Ξ is a convex compact set.

The MM $F(x) = f(x) + \Xi$ is strongly differentiable [12], and here $F'_+(x^0, \upsilon) = f'(x^0)\upsilon$, $F'_-(x^0, \upsilon) = 0$. It follows from Theorem 3.2 that the set X_0 is strongly asymptotically stable if the norm of the matrix $f'(x)$ is less than unity for all points $x \in \mathrm{bd}X_0$.

The use of Theorem 3.2 to investigate the stability of SSs of difference inclusions impinges on the substantial problems produced by the excessive restrictiveness of the strong differentiability condition on an MM. For example, this condition is not necessarily satisfied by an MM of the form

$$F(x) = \{Ax:\ A \in \Omega\} \tag{3.4}$$

where Ω is some convex set of $(n \times n)$-matrices. This effectively devalues Theorem 3.2 because difference inclusions with a right-hand side of the form (3.4) happens to be the subject of investigation in robust stability theory for linear discrete systems.

This difficulty can be overcome by relating the concept of the derivative of an MM to functions of the form (3.4). This enables one to justify a stronger assertion. The MMs (3.4) were considered in connection with the study of generalized Jacobians of non-smooth functions [13] and were given a generalization called a sheaf [14].

Definition 3.1 [15]. The MM $D(\cdot): R^n \to R^n$ is called a sheaf if

1. the sets $D(x)$ are non-empty, convex and compact;
2. $D(x^1 + x^2) \subset D(x^1) + D(x^2)$, $\forall x^1, x^2$;
3. $D(\lambda x) = \lambda D(x)$, $\forall x \in R^n$, $\lambda > 0$;
4. $\| D \| = \sup_{\|x\| \leq 1} \sup_{y \in D(x)} \| y \| < +\infty$.

The concept of a sheaf is used to define the prederivative of a non-smooth function [14, 15]. One can proceed similarly for an MM.

Definition 3.2. The sheaf $D(\cdot) = D^+F(x^0, \cdot)$ is called an upper prederivative of the MM $F(\cdot)$ at the point x^0 if

$$\lim_{\upsilon \to 0} \| \upsilon \|^{-1} h_+(F(x^0) + D(\upsilon),\quad F(x^0 + \upsilon)) = 0 \tag{3.5}$$

is satisfied.

The class of MMs which have upper prederivatives is fairly wide and includes, in particular, MMs which satisfy the Lipschitz condition. However, it is easy to give an example of a continuous MM for which (3.5) is not satisfied by any sheaf $D(\cdot)$. For example, $F(x) = [0, |x|^{1/2}]$ does not have an upper prederivative at the point $x = 0$.

One can formulate the following extension of Theorem 2.1 in terms of upper prederivatives.

Theorem 3.3. Let X_0 be a compact set which is invariant under (1.2), and let $F(\cdot)$ be a continuous MM which has a prederivative $D^+F(\cdot, \cdot)$ continuous in the neighbourhood of $(\mathrm{bd}X_0) \times S$. If for all $x \in \mathrm{bd}X_0$ $\| D^+F(x, \cdot) \| < 1$, then X_0 is a strongly asymptotically stable set with respect to (1.2).

The proof of this theorem is exactly the same as the proof of Theorem 3.2, and will not be given here.

Example 3.2. Suppose that X_0 is an invariant (stationary) set with respect to the different inclusion

$$x_{i+1} \in D(x_i) + \Xi \tag{3.6}$$

where $D(\cdot)$ is an MM of the form (3.4). It is obvious that at every point x the MM $F(x) = D(x) + \Xi$ has an upper prederivative which is identical with $D(\cdot)$ and $\| D^+F(x, \cdot) \| = \| D(\cdot) \| = \max\{\| A \| : A \in \Omega\}$. The invariant set X_0 is thus strongly asymptotically stable with respect to (3.6) if $\| A \| < 1$ for all $A \in \Omega$.

Example 3.3. We consider the one-dimensional difference inclusion

$$x_{i+1} \in F(x_i), \quad F(x) = [-(\alpha x^2 + 1), (\alpha x^2 + 1)], \quad 0 < \alpha < 1/4$$

It can be shown that the intervals $X_0^1 = [-x^1, x^1]$ and $X_0^2 = [-x^2, x^2]$ where $x^{12} = (1 \pm \sqrt{(1-4\alpha)})/(2\alpha)$, are SSs of the system under consideration, and that the interval X_0^2 is a minimal SS. I can also be established that

$$D^+F(x, \upsilon) = [-2\alpha x\upsilon, 2\alpha x\upsilon], \quad \|D^+F(x, \cdot)\| = 2\alpha|x|$$

Hence, by Theorem 3.3, the set X_0^2 is strongly asymptotically stable. The norm of the upper prederivative at the boundary points of the integral X_0^i is greater than unity, and this set is not stable.

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REFERENCES

1. DZHURI E. I., The robustness of discrete systems. *Avtomat. Telemekh.* **5**, 3–28, 1990.
2. SILJAK D., Parameter-space methods for robust control design: a guided tour. *IEEE Trans. Automat. Control.* **AC-34,** 7, 674–688, 1989.
3. PSHENICHNYI B. N., *Convex Analysis and Extremal Problems*. Nauka, Moscow, 1980.
4. AUBIN J. P. and FRANKOWSKA H., *Set-valued Analysis*. Birkauser, Boston, 1990.
5. MASCHLER M. and PELEG B., Stable sets and stable points of set-valued dynamic systems with applications to game theory. *SIAM J. Control and Optimiz.* **14**, 6, 985–995, 1976.
6. KUNTSEVICH V. M. and CHEKHOVOI Yu. N., *Non-linear Control Systems with Pulse-frequency and Broad-band Modulation*. Tekhnika, Kiev, 1970.
7. KOLOMOGOROV A. N. and FOMIN S. V., *Elements of the Theory of Functions and Functional Analysis*. Nauka, Moscow, 1976.
8. COLLATI L., *Functional Analysis and Computational Mathematics*. Mir, Moscow, 1969.
9. KRASNOSEL'SKII M. A., VAINIKKO G. M., ZABREIKO P. P., RUTITSKII Ya. B. and STETSENKO V. Ya., *Approximate Solution of Operator Equations*. Nauka, Moscow, 1969.
10. SMIRNOV G. V., Weak asymptotic ability of differential inclusions in the first approximation. *Vychisl. Tsentr. Akad. Nauk SSSR*, Moscow, 1989.
11. TYRIN Yu. N., Mathematical formulation of a simplified model of industrial planning. *Ekon. Mat. Metody* **1**, 3, 391–409, 1965.
12. PECHERSKAYA N. A., On the differentiability of multivalued maps. *Vestn. Len. Gos. Univ.* **2**, 7, 115–117, 1981.
13. CLARK F., *Optimization and Non-smooth Analysis*. Nauka, Moscow, 1988.
14. IOFFE A. D., Non-smooth analysis: differential calculus of non-differentiable mapping. *Trans. Am. Math. Soc.* **266,** 1, 1–56, 1981.
15. DEM'YANOV V. F. and RUBINOV A. M., *Foundations of Non-smooth Analysis and Quasidifferential Calculus*. Nauka, Moscow, 1980.

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